WPI Mathematical Sciences Ph.D. General Comprehensive Exam

MA 540 Probability and Mathematical Statistics-I

May, 2018

Note: Please make sure to write down your thinking process in bullet points even if you cannot solve the problems perfectly.

1. (a) Let $f(u, v) = 1$ if $0 \leq u \leq e^{-v}, v \geq 0$ and $f(u, v) = 0$ otherwise. Find $f(u)$, $f(v)$, $f(u \mid v)$ and $f(v \mid u)$.

   (b) Let $X, Y \overset{ind}{\sim} \text{Normal}(0, 1)$. Suppose $X < Y$, find the joint pdf of X and Y. What is the joint pdf of X and Y if $X = Y$?

2. (a) Let $\log(X) \sim \text{Normal}(0, 1)$. Find the pdf of X and $E(X^k)$ for any integer k. Deduce the variance of X.

   (b) i. Find a that minimizes $E\{(X - a)^2\}$, where X is a random variable with finite variance. Does your value of a really exist? Explain.

   ii. Let $X_1, \ldots, X_n$ be independent with mean $a \neq 0$ and variance 1. Consider $T = \sum_{i=1}^{n} w_i X_i$, where $w_1, \ldots, w_n$ are unknown positive quantities. Suppose $E(T) = a$, find $w_1, \ldots, w_n$ that minimize $\text{Var}(T)$.

3. Let X be a Normal$(0, 1)$ random variable. Find the pdf or pmf of $Y = X^n$, where n is a non-negative integer.

4. Let $X_1, \ldots, X_n \overset{iid}{\sim} \text{Uniform}(0, 1)$. Let $X_{(1)}$ and $X_{(n)}$ be the smallest and largest order statistics. Show that $X_{(n)}$ and $1 - X_{(1)}$ converge almost surely to 1 as $n \to \infty$.

5. Let $X > 0$ be a random variable with its moment generation function $M(t)$. Show that for all real t such that $M(t)$ exists,

$$P(tX > \epsilon^2 + \log(M(t))) \leq e^{-\epsilon^2}.$$

   (Here log is the natural logarithm.)

6. Let $F(x)$ be the CDF of a random variable X. Define $Y = F(X)$ (i.e., the CDF transformation), show that if X is **discrete**, $P(Y \leq y) \leq y$ for any $y \in (0, 1)$, and $P(Y \leq y) < y$ for some $y \in (0, 1)$.